

Late time tails in the Kerr spacetime

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Abstract

Outside a black hole, perturbation fields die off in time as $1/t^n$. For spherical holes $n = 2\ell + 3$ where ℓ is the multipole index. In the nonspherical Kerr spacetime there is no coordinate-independent meaning of "multipole," and a common sense viewpoint is to set ℓ to the lowest radiatable index, although theoretical studies have led to very different claims. Numerical results, to date, have been controversial. Here we show that expansion for small Kerr spin parameter a leads to very definite numerical results confirming previous theoretical analyses.

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Perturbation fields outside a spherically symmetric black hole die off in time in a way that has been well understood for more than 35 years[1]. This understanding is closely tied to the fact that the spherical background allows the fields to be decomposed into multipoles, each of which evolves from initial data independently and can be studied with a relatively simple 1+1 computer code. The evolved radiation field starts with oscillations characteristic of the details of the initial data, then undergoes an epoch of quasinormal ringing, and lastly falls off in time t in the form of a late-time "tail" $t^{-2\ell-3}$, where ℓ is the multipole index[2].

For the nonspherical Kerr black hole the situation has been anything but clear since Kerr perturbations cannot be separated into independently evolving multipoles[3]. Certain symmetries do apply, however. Perturbations can be separated into nonmixing azimuthal Fourier modes $e^{im\phi}$ and into modes even or odd with respect to reflection through the equatorial plane. This has given rise to what we shall call a "common sense" viewpoint in which approximate spherical symmetry applies to the distant radiation field, and the late-time behavior of the evolution of initial data is dominated by the lowest multipole index (i.e., the most slowly dying tail) compatible with the azimuthal and equatorial symmetries of the initial data. Thus, for example, a scalar perturbation field whose initial data has $m = 0$ and that is symmetric with respect to the equator will, at late time, be predominately a monopole and will have the t^{-3} tail of a monopole.

Supporting this viewpoint is the fact that, without spherical symmetry in the background, there are no preferred angular coordinates. The radial r and polar θ coordinates are mixed differently in different systems of coordinates used to describe the Kerr spacetime, such as Boyer-Lindquist (BL) coordinates[4] or Kerr coordinates[5]. A "multipole" is specific to the coordinate choice, and therefore cannot determine a physical effect, like the rate of decrease of the field.

Theoretical work has argued against the common sense viewpoint, and claims have appeared of numerical results to support both sides of the argument. The argument for something other than "common sense," was first given by Hod[6, 7, 8, 9, 10], (see also Barack and Ori[11, 12]). Hod considered initial data that has only a single multipole $Y_{\ell m}$ in BL coordinates. By looking at the zero frequency limit of a Fourier transform, Hod argued that the tails of a massless scalar function would have the following dependence on time and on multipolarity of the initial data:

$$\Psi \propto \begin{cases} Y_{\ell m}/t^{2\ell+3} & \ell = m \text{ or } \ell = m + 1 \\ Y_{mm}/t^{\ell+m+1} & \ell - m \geq 2 \text{ (even)} \\ Y_{m+1\ m}/t^{\ell+m+2} & \ell - m \geq 2 \text{ (odd)}. \end{cases} \quad (1)$$

The common sense results seemed so compelling, that numerical work was immediately sought that would settle the issue, but numerical tests required rather delicate 2+1 codes. Krivan[13] was the first to attempt this work, using a scalar field with an initial outgoing pulse with BL multipole indices $\ell, m = 4, 0$. The Eq. (1) prediction for this case

is a t^{-5} monopole tail while the common sense prediction is a t^{-3} monopole. (This $\ell, m = 4, 0$ case is the simplest scalar case for which there are controversial predications, and we will consider it here as the primary test case.) Krivan's results weakly suggested a $t^{-5.5}$ law, but Krivan pointed out serious numerical problems caused by angular differencing. Subsequent studies of the $\ell, m = 4, 0$ case, by Burko and Khanna[14], and by Scheel *et al.*[15] both found the common sense t^{-3} result. But in reference [14] penetrating Kerr coordinates were used and in [15] Kerr-Schild coordinates were used. A constant time initial hypersurface in either of these systems is different from a hypersurface of constant BL time, and therefore neither of these studies represented the same problem as that to which Eq. (1) and Krivan's results apply.

A new element was added to the unresolved issues when Poisson published a careful analysis[16] of tail behavior within linearized general relativity. Since tails appear to develop in the weakly curved background far from a central black hole, and to be insensitive to the details of the strongly curved inner region[17], it seemed reasonable that Poisson's analysis would be relevant to the question of tails in Kerr spacetime, at least as an approximation for tails that evolve from initially outgoing pulses that start far from a central hole. Poisson's analysis, fundamentally different from Hod's, led to precisely the results in Eq. (1).

We present here a new approach for computationally probing the late time evolution of tails in the Kerr spacetime and, in principle, in other nonspherical spacetimes. The advantages of this approach are: (i) it gives a clear meaning to "multipoles" since it uses a spherical operator for evolution; (ii) there is no angular differencing, and hence it avoids the errors pointed out by Krivan; (iii) the method gives convergent, clear numerical answers to the controversies of tails in the Kerr spacetime. The new approach expands fields and the equations that govern them in powers of the spin parameter a , of the Kerr metric.

We show here the approach as applied to a scalar field ψ , both for simplicity, and because previous numerical work has all been for a scalar field. We make a further, minor simplification by choosing the scalar field to have no ϕ (azimuthal) dependence. More details and the case for more general fields will be published elsewhere[18].

In Boyer-Lindquist coordinates, the Teukolsky equation for this case (equivalent to $\psi_{,\alpha}^{\alpha} = 0$) takes the explicit form

$$L[\psi] \equiv \left[\frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta \right] \frac{\partial^2 \psi}{\partial t^2} - \Delta \frac{\partial^2 \psi}{\partial r^2} - 2(r - M) \frac{\partial \psi}{\partial r} - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) = 0 \quad (2)$$

where Δ has the usual meaning $r^2 - 2Mr + a^2$, with M the mass and $a = J/M$ the angular momentum parameter. Units are used in which $c = G = 1$.

For any initial data, the field ψ that evolves will depend on the spin parameter a and we expand both ψ and the operator of Eq. (2) in powers of a/M :

$$\psi = \psi^{(0)} + a/M \psi^{(1)} + (a/M)^2 \psi^{(2)} + \dots \quad L = L^{(0)} + (a/M)^2 L^{(2)} + \dots \quad (3)$$

Here the $\psi^{(n)}$ are functions of t, r and θ . The equations that result for the even powers of a/M are

$$L^{(0)}[\psi^{(0)}] = 0 \quad (4)$$

$$L^{(0)}[\psi^{(2)}] = -L^{(2)}[\psi^{(0)}] \quad (5)$$

$$L^{(0)}[\psi^{(4)}] = -L^{(4)}[\psi^{(0)}] - L^{(2)}[\psi^{(2)}] \quad (6)$$

and so forth.

The evolved fields can be solved order by order. Since the right hand sides in the above sequence are treated as known driving terms, only the operator $L^{(0)}$ need be inverted, but $L^{(0)}$ is just the spherically symmetric Schwarzschild operator, so the multipoles contained in the solutions will only be those that appear in the driving terms.

We focus now on the primary test case $\ell = 4$, and on the question of the exponent for the late time tail. Since the initial data has $\ell = 4$ there will be an evolved field that is zero order in a/M , i.e., the purely $\ell = 4$ field that is evolved by the Schwarzschild operator $L^{(0)}$. The $\ell = 4$ zeroth order field $\psi^{(0)}$ will provide a source term on the right hand side of Eq. (5). But $L^{(2)}$ (the only term in the full set of $L^{(n)}$ that is not spherically symmetric) contains a term $-M^2 \sin^2 \theta \partial_t^2$, and

$$\sin^2 \theta P_4(\cos \theta) = -(10/33) P_6(\cos \theta) + (38/77) P_4(\cos \theta) - (4/21) P_2(\cos \theta).$$

The driving term in Eq. (5) will then have a quadrupole piece, and hence $\psi^{(2)}$, the solution of Eq. (5), will contain a quadrupole. In a similar manner, the second-order quadrupole will provide a fourth-order (in a/M) monopole source term in Eq. (6). The fourth-order field $\psi^{(4)}$ will therefore have a monopole, and hence a monopole late-time tail.

Paying attention only to the multipoles of interest for this late-time tail, we write the relevant multipoles of the relevant orders as follows:

$$\psi^{(0)}(t, r, \theta) = r^{-1} f_4^{(0)}(t, r) P_4(\cos \theta) \quad \psi^{(2)}(t, r, \theta) = r^{-1} f_2^{(2)}(t, r) P_2(\cos \theta) \quad \psi^{(4)}(t, r, \theta) = r^{-1} f_0^{(4)}(t, r) P_0(\cos \theta) \quad (7)$$

and the order-by-order 1+1 wave equations of interest become

$$\partial_t^2 f_4^{(0)} - \partial_{r^*}^2 f_4^{(0)} + \frac{1 - 2M/r}{r^2} \left(20 + \frac{2M}{r} \right) f_4^{(0)} = 0 \quad (8)$$

$$\partial_t^2 f_2^{(2)} - \partial_{r^*}^2 f_2^{(2)} + \frac{1 - 2M/r}{r^2} \left(6 + \frac{2M}{r} \right) f_2^{(2)} = -\frac{4}{21} \frac{M^2}{r^2} \left(1 - \frac{2M}{r} \right) \partial_t^2 f_4^{(0)} \quad (9)$$

$$\partial_t^2 f_0^{(4)} - \partial_{r^*}^2 f_0^{(4)} + \frac{1 - 2M/r}{r^2} \left(\frac{2M}{r} \right) f_0^{(4)} = -\frac{2}{15} \frac{M^2}{r^2} \left(1 - \frac{2M}{r} \right) \partial_t^2 f_2^{(2)}. \quad (10)$$

The numerical computations were carried out on a t, r^* characteristic grid, where r^* is the standard Schwarzschild tortoise coordinate, with $\Delta t = \Delta r^*$, and with no boundary conditions. (Computations were carried out only in the domain of dependence of the initial spatial grid.) All three fields $f_4^{(0)}, f_2^{(2)}, f_0^{(4)}$ were evolved simultaneously. Initial data for $f_4^{(0)}$, at $t = 0$, was chosen to be a Gaussian pulse and was made (approximately) outgoing by taking the initial value at grid point r^* to be replicated after a time step Δt , at the spatial grid point $r^* + \Delta t$. Initial data were taken to be zero for $f_2^{(2)}$ and for $f_0^{(4)}$, so that the final fourth-order monopole was only the result of the initial pure $\ell = 4$ data.

Typical results are shown in Fig. 1. At earliest times these curves show the nonzero starting value of the zeroth order field $f_4^{(0)}$, the field for which there is nonzero initial data. This field then “ignites” the second-order quadrupole which in turn ignites the fourth-order monopole. All fields go through periods of quasinormal ringing (each with a different frequency) around $t/2M = 100$ and settle into the power-law tails $1/t^n$ that show up as straight lines in the log-log plots. With better than 1% accuracy, we find that $f_4^{(0)}$ has an exponent $n = 11$, and $f_2^{(2)}$ has $n = 7$, values that are in excellent agreement with the $n = 2\ell + 3$ rule and that demonstrate that each tail is evolving without significant influence from the lower-order tail that, at earlier times, “ignited” it.

Of particular interest, of course, is the exponent of the monopole tail, which we found to be $n = 5$ to high accuracy (better than 1%). We ran this case for several grid sizes. For Δr^* less than $0.025(2M)$ we found some differences of as much as 5% in the early waveform details, but negligible difference in the computed value of n . The major source of error lay in determining the asymptotic value of the exponent. To do this we ran until times of about $t = 4000(2M)$ with runs that required around a day on a workstation. Exponent n was computed for segments of the late-time tail, and was found to be 5.0495 for the latest segment. The results showed clear evidence that the value of n was slowly asymptoting to a slightly smaller value. We do not consider it plausible that at sufficiently late times it would decrease to $n = 3$.

From a numerical point of view this is truly remarkable. Since the fields at late time are evolving without source (as shown by the success of the $2\ell + 3$ rule for the lower order multipoles), the monopole *should* have the exponent $n = 3$. Indeed, modifications in the computation do change the exponent from $n = 5$ to $n = 3$. We see this change if we put in nonzero initial data for $f_0^{(4)}$, or for $f_2^{(2)}$. We see this change also if we arbitrarily turn off the evolution of the $f_4^{(0)}$ field at some intermediate time and let the $f_2^{(2)}$ and $f_0^{(4)}$ fields continue to evolve. The overwhelming tendency for the exponent to be 3 rather than 5 convinces us that there is no error we have overlooked in our program; any error would almost surely lead to $n = 3$.

The delicacy of the the $n = 5$ result underscores the numerical advantages of the 1+1 computations in Eqs. (8) – (10) over 2+1 codes, even though the set of equations contains second time derivatives applied twice on the right hand source terms. The method turns out to be accurate enough that we have been able to go one step further, and treat the initial $\ell, m = 6, 0$ case for which the common sense monopole exponent is $n = 3$, while Eqs. (1) predicts $n = 7$. We have found $n = 7$, again to within about 1%.

There remains the possibility that the $n = 5$ result is a consequence of cancellation of contributions to the $1/t^3$ tail, and that these cancellations only occur to lowest order in a/M (that is, to order $(a/M)^4$). In a forthcoming paper[18] we shall present the results carried out to higher order. We shall also provide further details of the method used, and a wider set of numerical examples, including those for nonaxisymmetric initial data and for gravitational perturbations, and tests of Poisson’s approximation[16].

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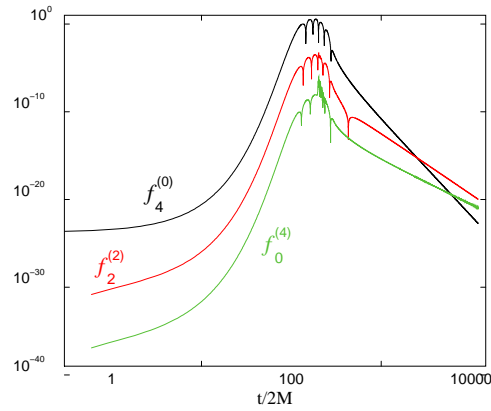


FIG. 1: Evolution of the zeroth-order $\ell = 4$ multipole $f_4^{(0)}$, the second-order $\ell = 2$ multipole $f_2^{(2)}$, and the the zeroth-order monopole $f_0^{(4)}$ for a scalar field in a Kerr spacetime.

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